

COMPUTATION OF ACOUSTIC FIELD ON 2D FRONTS PART 1. TRAJECTORIES

NICK MALTSEV

ABSTRACT. Traditional methods of wave field computation include modes, rays and parabolic approximation. This report presents description of new approach, based on computation of wave fronts and corresponding phases and amplitudes of the field on them. Final result can be interpreted as sum of ray-like arrivals, computed while running fronts intersect receiver location, and contain arrival times and angles. Introduction of involutes of the wave fronts clarify the structure of the field. Examples of application of this method are presented.

Keywords: Rays, fronts, asymptotic expansion, curvature.

1. RAY EQUATIONS

In 2D ray equations are [1]:

$$(1.1) \quad \frac{d\mathbf{r}}{d\tau} = c(\mathbf{r})\mathbf{t},$$

$$(1.2) \quad \frac{d\theta}{d\tau} = -(\nabla c, \mathbf{n}),$$

$$(1.3) \quad \mathbf{t} = \mathbf{i} \cos \theta + \mathbf{k} \sin \theta, \mathbf{n} = -\mathbf{i} \sin \theta + \mathbf{k} \cos \theta$$

$$\mathbf{r}(0) = \mathbf{r}_0, \theta(0) = \alpha$$

where α is a launch angle in the source. Parameters α, τ are *ray coordinates*. Same parameters for our ray coordinates are

$$(1.4) \quad h_\tau = \sqrt{x_\tau^2 + z_\tau^2} = c, h_\alpha = \sqrt{x_\alpha^2 + z_\alpha^2} = J$$

so

$$(1.5) \quad \nabla = \frac{\mathbf{t}}{h_\tau} \frac{\partial}{\partial \tau} + \frac{\mathbf{n}}{h_\alpha} \frac{\partial}{\partial \alpha}$$

Wave front defined as a curve $\tau = \tau_0$ or

$$(1.6) \quad x = x(\tau_0, \alpha), z = z(\tau_0, \alpha), \theta = \theta(\tau_0, \alpha)$$

Vector tangent to the front is

$$(1.7) \quad \mathbf{r}_\alpha(\tau_0, \alpha) = \mathbf{i}x_\alpha + \mathbf{j}z_\alpha \equiv \mathbf{n}J = J(-\mathbf{i} \sin \theta + \mathbf{k} \cos \theta)$$

so

$$(1.8) \quad x_\alpha = -J \sin \theta, z_\alpha = J \cos \theta, \Theta = \theta_\alpha$$

To compute the curvature of the front κ we need

$$(1.9) \quad x_{\alpha\alpha} = -J_\alpha \sin \theta - J\Theta \cos \theta, z_{\alpha\alpha} = J_\alpha \cos \theta - J\Theta \sin \theta$$

then

$$(1.10) \quad \kappa = -\frac{x_{\alpha\alpha}z_{\alpha} - z_{\alpha\alpha}x_{\alpha}}{J^3} = \frac{\Theta}{J}$$

and radius of the curvature of the front

$$(1.11) \quad \mathcal{R}(\alpha, \tau) = \frac{1}{\kappa} = \frac{J(\alpha, \tau)}{\Theta(\alpha, \tau)}$$

We also calculate

$$J_{\tau} = \frac{\partial}{\partial \tau} \sqrt{x_{\alpha}^2 + z_{\alpha}^2} = J^{-1}(x_{\alpha}x_{\alpha\tau} + z_{\alpha}z_{\alpha\tau}) = \\ -\sin \theta (J(\nabla c, \mathbf{n}) \cos \theta - c \sin \theta \Theta) + \cos \theta (J(\nabla c, \mathbf{n}) \sin \theta + c \cos \theta \Theta) = c\Theta$$

so

$$(1.12) \quad \mathcal{R}(\alpha, \tau) = \frac{cJ}{J_{\tau}}$$

2. FRONT EQUATIONS

Consider wave fronts as an involutes[2] of some curve \mathbf{C} , or

$$(2.1) \quad \mathbf{r}(\alpha, \tau) = \mathbf{C}(\alpha, \tau) + \mathcal{R}(\alpha, \tau)\mathbf{t}(\alpha, \tau)$$

where $\mathbf{C} = \mathbf{i}X_c + \mathbf{k}Z_c$ is some unknown center, \mathcal{R} radius of front curvature and \mathbf{t} - unit tangent vector along the ray. Differentiating last equation with respect to α we obtain

$$\mathbf{r}_{\alpha} = \mathbf{n}J = \mathbf{C}_{\alpha} + \mathcal{R}_{\alpha}\mathbf{t} + \mathcal{R}\mathbf{n}\theta_{\alpha}$$

or

$$(2.2) \quad \mathbf{C}_{\alpha} + \mathcal{R}_{\alpha}\mathbf{t} = 0$$

Differentiating equation (2.1) with respect to τ we have

$$(2.3) \quad \mathbf{r}_{\tau} = \mathbf{t}c = \mathbf{C}_{\tau} + \mathcal{R}_{\tau}\mathbf{t} + \mathcal{R}\mathbf{n}\theta_{\tau}$$

or

$$(2.4) \quad \mathbf{C}_{\tau} + (\mathcal{R}_{\tau} - c)\mathbf{t} - \mathcal{R}\mathbf{n}(\nabla c, \mathbf{n}) = 0$$

In free space, when $c = \text{const}$

$$(2.5) \quad \mathbf{C}_{\tau} + (\mathcal{R}_{\tau} - c)\mathbf{t} = 0, \mathbf{C}_{\tau} = 0, \mathcal{R}_{\tau} = c,$$

where the last equation is an illustration of Huygens principle. When we have media with constant speed of sound between some boundaries, last equations obviously produce location of "imaginary" or "reflected" sources, \mathbf{C} , which generate field propagating with speed c . So method of reflections appears to be an intrinsic part of ray equations. In general case

$$\mathcal{R}_{\tau} = \frac{dJ}{d\tau} \frac{1}{\Theta} = \frac{J_{\tau}\Theta - J\Theta_{\tau}}{\Theta^2} = c - \mathcal{R} \frac{\Theta_{\tau}}{\Theta}$$

so

$$(2.6) \quad \mathbf{C}_{\tau} = -(\mathcal{R}_{\tau} - c)\mathbf{t} + \mathcal{R}\mathbf{n}(\nabla c, \mathbf{n}) = 0$$

$$(2.7) \quad \mathbf{C}_{\tau} = \mathcal{R} \frac{\Theta_{\tau}}{\Theta} \mathbf{t} + \mathcal{R}\mathbf{n}(\nabla c, \mathbf{n})$$

$$(2.8) \quad \mathbf{C}_{\tau} = \mathcal{R} \left(\frac{\Theta_{\tau}}{\Theta} \mathbf{t} + (\nabla c, \mathbf{n}) \mathbf{n} \right)$$

