

FORMAL SOLUTION OF SCALAR RICCATI EQUATION

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ABSTRACT. Formal solution of Riccati equation is received.

1. SCALAR RICCATI EQUATION

General Riccati equation[1]

$$(1.1) \quad \frac{dy(x)}{dx} = a(x)y^2(x) + b(x)y(x) + c(x)$$

has solution in a form of infinite procedure

$$(1.2) \quad y(x) = \frac{1}{2a}(-b \pm \sqrt{b^2 - 4a(c - \frac{d}{dx}(\frac{1}{2a}(-b \pm \sqrt{b^2 - 4a(c - \frac{d}{dx}(\frac{1}{2a}(-b \pm \sqrt{b^2 - 4a(c - \frac{d}{dx}(\dots))))))))))})$$

Direct substitution of $y(x)$ from Eq. 1.2 to Eq. 1.1 transforms last to identity:

$$\frac{1}{4a}(b^2 \mp 2b\sqrt{b^2 - 4a(c - \frac{dy}{dx})} + b^2 - 4a(c - \frac{dy}{dx})) + \frac{b}{2a}(-b \pm \sqrt{b^2 - 4a(c - \frac{dy}{dx})}) + c = \frac{dy}{dx}$$

The procedure described above can be expressed in terms of iterations:

$$y_0(x) = \frac{1}{2a(x)}[-b(x) \pm \sqrt{b^2(x) - 4a(x)c(x)}]$$
$$y_{n+1}(x) = \frac{1}{2a(x)}[-b(x) \pm \sqrt{b^2(x) - 4a(x)(c(x) - \frac{dy_n}{dx})}], n \geq 0$$

Example of this approach applied to the receiving of approximate solutions of Airy equation[2] in the following section.

2. AIRY EQUATION

Solutions of the Airy equation

$$(2.1) \quad \frac{d^2y(x)}{dx^2} + xy(x) = 0$$

can be looked in the form

$$(2.2) \quad y(x) = \exp(i \int^x q(x) dx)$$

and for $q(x)$ it becomes Riccati equation

$$(2.3) \quad q^2(x) - x = i \frac{dq(x)}{dx}$$

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According to Eq. 1.2 formal solution of this equation has the form

$$(2.4) \quad q(x) = \pm \sqrt{x \pm i \frac{d}{dx} \sqrt{x \pm i \frac{d}{dx} \sqrt{x \dots}}}$$

If to take first two iterations

$$\begin{aligned} q_0(x) &= \pm x^{0.5} \\ q_1(x) &= \pm \sqrt{x \pm \frac{i}{2} x^{-0.5}} = \pm x^{-0.25} \sqrt{x^{1.5} \pm \frac{i}{2}} \end{aligned}$$

Then

$$\begin{aligned} I_0^{(1,2)}(x) &= \int^x q_0(t) dt = \pm \frac{2}{3} x^{\frac{3}{2}} \\ I_1^{(1,2)}(x) &= \int^x q_1(t) dt = \pm \frac{2}{3} x^{\frac{3}{4}} \sqrt{x^{\frac{3}{2}} \pm \frac{i}{2}} + \frac{i}{3} \ln(x^{\frac{3}{4}} + \sqrt{x^{\frac{3}{2}} \pm \frac{i}{2}}) \\ I_1^{(1,2)}(0) &= \frac{i}{3} \ln(\sqrt{\pm \frac{i}{2}}) = \frac{i}{6} \ln(\pm \frac{i}{2}) = \frac{i}{6} (\pm i \frac{\pi}{2} - \ln 2) = \mp \frac{\pi}{12} - i \frac{\ln 2}{6} \end{aligned}$$

According to [2] first two terms of asymptotic expansion for Airy functions when $\Re x \rightarrow \infty, \Im x = 0$ are:

$$\begin{aligned} Ai(x) &\sim \pi^{-\frac{1}{4}} x^{-\frac{1}{4}} \sin\left(\frac{2}{3} x^{\frac{3}{2}} + \frac{\pi}{4}\right) \\ &= \frac{1}{i\pi^{\frac{1}{4}}} (\exp(i(\frac{2}{3} x^{\frac{3}{2}} + \frac{\pi}{4})) - \frac{1}{4} \ln x - \ln 2) - \exp(-i(\frac{2}{3} x^{\frac{3}{2}} + \frac{\pi}{4})) - \frac{1}{4} \ln x - \ln 2) \\ Bi(x) &\sim \pi^{-\frac{1}{4}} x^{-\frac{1}{4}} \cos\left(\frac{2}{3} x^{\frac{3}{2}} + \frac{\pi}{4}\right) \\ &= \frac{1}{\pi^{\frac{1}{4}}} (\exp(i(\frac{2}{3} x^{\frac{3}{2}} + \frac{\pi}{4})) - \frac{1}{4} \ln x - \ln 2) + \exp(-i(\frac{2}{3} x^{\frac{3}{2}} + \frac{\pi}{4})) - \frac{1}{4} \ln x - \ln 2) \end{aligned}$$

When $\Re x \rightarrow -\infty, \Im x = 0$

$$\begin{aligned} Ai(x) &\sim \pi^{-\frac{1}{2}} \exp(-\frac{2}{3}(-x)^{\frac{3}{2}} - \frac{1}{4} \ln(-x)) \\ Bi(x) &\sim \pi^{-\frac{1}{4}} \exp(\frac{2}{3}(-x)^{\frac{3}{2}} - \frac{1}{4} \ln(-x)) \end{aligned}$$

When $\Re x \rightarrow \infty, \Im x = 0$

$$(2.5) \quad iI_1^{(1,2)}(x) \sim \pm i \frac{2}{3} x^{\frac{3}{2}} - \frac{\ln 2}{3} - \frac{1}{4} \ln x$$

When $\Re x \rightarrow -\infty, \Im x = 0$

$$(2.6) \quad iI_1^{(1,2)}(x) \sim \mp \frac{2}{3}(-x)^{\frac{3}{2}} - \frac{\ln 2}{3} - \frac{\pi}{3} - \frac{1}{4} \ln(-x)$$

Comparing Eq. 2.5-2.6 with asymptotic expansions we receive approximate expressions for Airy functions for $\Re x \geq 0, \Im x = 0$:

$$(2.7) \quad Ai(x) \sim \frac{1}{i\pi^{\frac{1}{4}} 2^{\frac{2}{3}}} (\exp(i(I_1^{(1)}(x) + \frac{\pi}{4})) - \exp(i(I_1^{(2)}(x) - \frac{\pi}{4})))$$

$$(2.8) \quad Bi(x) \sim \frac{1}{\pi^{\frac{1}{4}} 2^{\frac{2}{3}}} (\exp(i(I_1^{(1)}(x) + \frac{\pi}{4})) + \exp(i(I_2^{(1)}(x) - \frac{\pi}{4})))$$

and for $\Re x \leq 0, \Im x = 0$

$$(2.9) \quad Ai(x) \sim \sin \frac{\pi}{6} \frac{2^{\frac{1}{3}}}{\pi^{\frac{1}{4}}} \Re \exp(i(I_1^{(1)}(x) + \frac{\pi}{12}))$$

$$(2.10) \quad Bi(x) \sim \cos \frac{\pi}{6} \frac{2^{\frac{1}{3}}}{\pi^{\frac{1}{4}}} \Re \exp(-i(I_1^{(1)}(x) + \frac{\pi}{12}))$$

(2.11)

At Fig.1 approximate values of $Ai(x), Bi(x)$ are displayed. They have discontinuity of first derivative at $x = 0$, while traditional asymptotic expressions are singular at $x = 0$.

3. APPENDIX A

For computation of integral

$$I_1(x) = \int^x q_1(t) dt = \pm \int^x \sqrt{t \pm \frac{i}{2} t^{-0.5}} dt$$

we introduce new independent variable

$$y = t^{\frac{3}{4}}, t = y^{\frac{4}{3}}, dt = \frac{4}{3} y^{\frac{1}{3}} dy$$

and

$$\begin{aligned} I_1(x) &= \pm \frac{4}{3} \int^x \sqrt{y^{\frac{4}{3}} \pm \frac{i}{2} y^{-\frac{2}{3}} y^{\frac{1}{3}}} dy = \pm \frac{4}{3} \int^x \sqrt{y^2 \pm \frac{i}{2}} dy = \\ &\pm \frac{4}{3} \left(\frac{1}{2} y \sqrt{y^2 \pm \frac{i}{2}} \pm \frac{i}{4} \ln(y + \sqrt{y^2 \pm \frac{i}{2}}) \right) \Big|_0^x = \\ &\pm \frac{2}{3} \left(x^{\frac{3}{4}} \sqrt{x^{\frac{3}{2}} \pm \frac{i}{2}} \pm \frac{i}{2} \ln(x^{\frac{3}{4}} + \sqrt{x^{\frac{3}{2}} \pm \frac{i}{2}}) \right) \end{aligned}$$

REFERENCES

1. Agnew, R.P.,: Differential Equations, McGraw-Hill, NY, 1960
2. Abramowitz, M., Stegun I.A., Handbook of mathematical functions, 1964

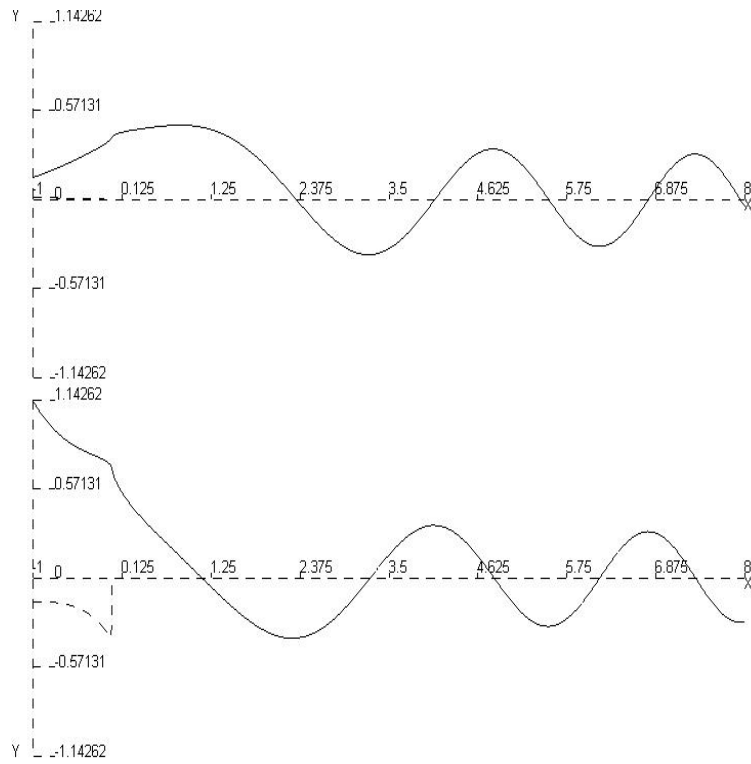


FIGURE 1. Approximate solutions of Airy equation