How many equators have 5-dimensional sphere?

Nikolai E. Maltsev
San Jose, CA

1 3-dimensional sphere

Equation of the *body* of the sphere in 3-D Euclidean space (x_1, x_2, x_3) is [1]:

$$x_1^2 + x_2^2 + x_3^2 \leq R^2 \tag{1}$$

It is a 3-D object. A *surface* of 3-D sphere

$$x_1^2 + x_2^2 + x_3^2 = R^2 \tag{2}$$

is a 2-dimensional, since x_1, x_2, x_3 are tied with equation (2), so only two of them are independent. It especially can be seen from parametric representation of sphere surface as

$$\begin{aligned} x_1 &= R\cos\alpha_0\cos\alpha_1 && (3)\\ x_2 &= R\cos\alpha_0\sin\alpha_1 && (4)\\ x_3 &= R\sin\alpha_0 && (5)\\ -\frac{\pi}{2} &< \alpha_0 < \frac{\pi}{2}, 0 \leq \alpha_1 < 2\pi, && (6) \end{aligned}$$

where α_0 is, of course,*a latitude*, and α_1 is *a longitude* . If $\alpha_0 = 0$ then a circle in plane (x_1, x_2)

$$\begin{aligned} x_1 &= R\cos\alpha_1\\ x_2 &= R\sin\alpha_1\\ x_3 &= 0\\ 0 &\leq \alpha_1 < 2\pi, \end{aligned}$$

is a 1-dimensional *equator* of 3-dimensional sphere. When $\alpha_0 = -\frac{\pi}{2}$

$$\begin{aligned} x_1 &= 0\\ x_2 &= 0\\ x_3 &= -R \end{aligned}$$

we have *south pole* and When $\alpha_0 = \frac{\pi}{2}$

$$\begin{aligned} x_1 &= 0 \\ x_2 &= 0 \\ x_3 &= R \end{aligned}$$

it is, of course *a north pole*. When, at last, $\alpha_1 = 0$, a big half circle

$$\begin{aligned} x_1 &= R\cos\alpha_0 \\ x_2 &= 0 \\ x_3 &= R\sin\alpha_0 \\ -\frac{\pi}{2} &< \alpha_0 < \frac{\pi}{2} \end{aligned}$$

in the plane (x_1, x_3) is *a zero meridian* , named Greenwich meridian on our earth. Intersection of the meridian with equator $(\alpha_0 = 0, \alpha_1 = 0)$ is a single point

$$x_1 = R, x_2 = x_3 = 0$$

Last expression for meridian can be obtained by rotation of one point around axis x_2:

$$\begin{vmatrix} R\cos\alpha_0 \\ 0 \\ R\sin\alpha_0 \end{vmatrix} = \begin{vmatrix} \cos\alpha_0, & 0, & -\sin\alpha_0 \\ 0, & 1, & 0 \\ \sin\alpha_0, & 0, & \cos\alpha_0 \end{vmatrix} \begin{vmatrix} R \\ 0 \\ 0 \end{vmatrix}$$

Finally equation of 3-D sphere can be obtained by rotation of single point around axis x_2 to receive zero meridian and by rotation of zero meridian around axis x_3:

$$\begin{vmatrix} \cos\alpha_1, & -\sin\alpha_1, & 0 \\ \sin\alpha_1, & \cos\alpha_1, & 0 \\ 0, & 0, & 1 \end{vmatrix} \begin{vmatrix} \cos\alpha_0, & 0, & -\sin\alpha_0 \\ 0, & 1, & 0 \\ \sin\alpha_0, & 0, & \cos\alpha_0 \end{vmatrix} \begin{vmatrix} R \\ 0 \\ 0 \end{vmatrix} = \begin{vmatrix} R\cos\alpha_0\cos\alpha_1 \\ R\cos\alpha_0\sin\alpha_1 \\ R\sin\alpha_0 \end{vmatrix} \tag{7}$$

Result is -to receive a 2-D surface of 3-D sphere we must take a single point and perform 2 rotations, around 2 axes. Result is identical to initial Eq.(3-5).

Let us ascend at one more space dimension.

2 4-dimensional sphere

Equation of the *body* of the sphere in 4-D Euclidean space (x_1, x_2, x_3, x_4) is:

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 \leq R^2 \tag{8}$$

It is a 4-D object. A *surface* of 4-D sphere

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = R^2 \tag{9}$$

is a 3-dimensional, since x_1, x_2, x_3, x_4 are tied by equation (9), so only three of them are independent. It especially can be seen from parametric representation of sphere surface as

$$\begin{aligned}
x_1 &= R\cos\alpha_0\cos\alpha_1\cos\alpha_2 && (10)\\
x_2 &= R\cos\alpha_0\cos\alpha_1\sin\alpha_2 && (11)\\
x_3 &= R\cos\alpha_0\sin\alpha_1 && (12)\\
x_4 &= R\sin\alpha_0 && (13)\\
-\frac{\pi}{2} &< \alpha_0, \alpha_1 < \frac{\pi}{2}, 0 \leq \alpha_2 < 2\pi && (14)
\end{aligned}$$

where α_0, α_1 are *two latitudes* and α_2 is *a longitude*. We can see all 2-D projections of 4-D sphere at Figure 1. Number of projections it is the number of combinations from four elements by two:

$$C_4^2 = \frac{4 \times 3}{2} = 6$$

Numbers in top-left and right-bottom corners of projections correspond to the coordinate axes x_1, x_2, x_3, x_4.

If $\alpha_0 = 0$ then

$$\begin{aligned}
x_1 &= R\cos\alpha_1\cos\alpha_2\\
x_2 &= R\cos\alpha_1\sin\alpha_2\\
x_3 &= R\sin\alpha_1\\
x_4 &= 0\\
-\frac{\pi}{2} &< \alpha_1 < \frac{\pi}{2}, 0 \leq \alpha_2 < 2\pi
\end{aligned}$$

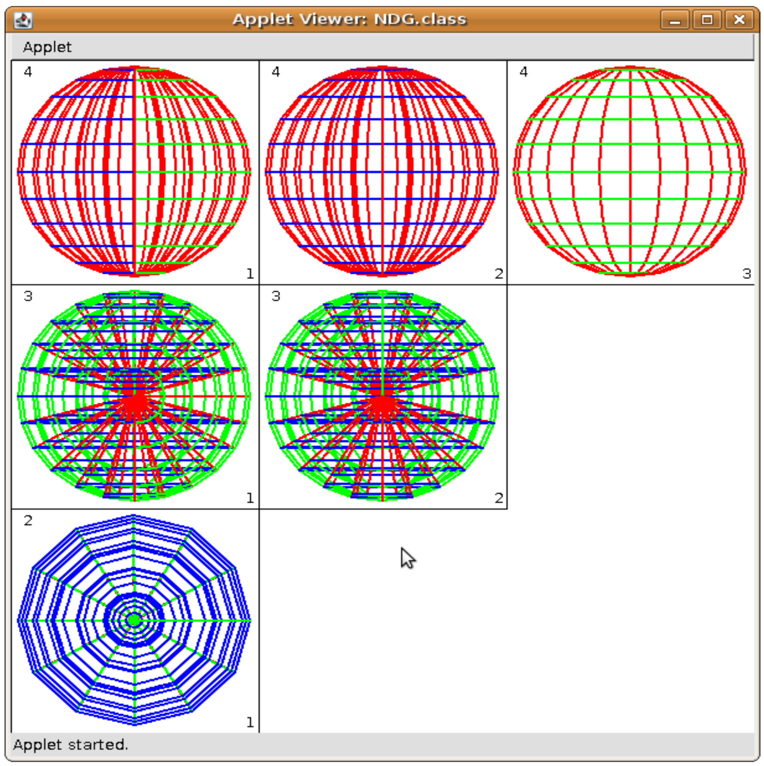

Figure 1: 4D Sphere. Red lines correspond to the change of α_0, green lines correspond to the change of α_1, blue lines correspond to the change of α_2,

is a 2-dimensional surface of 3-dimensional sphere, first *equator*, of 4-dimensional sphere in *hyperplane* (x_1, x_2, x_3).We can see this object og Figure 3. Term

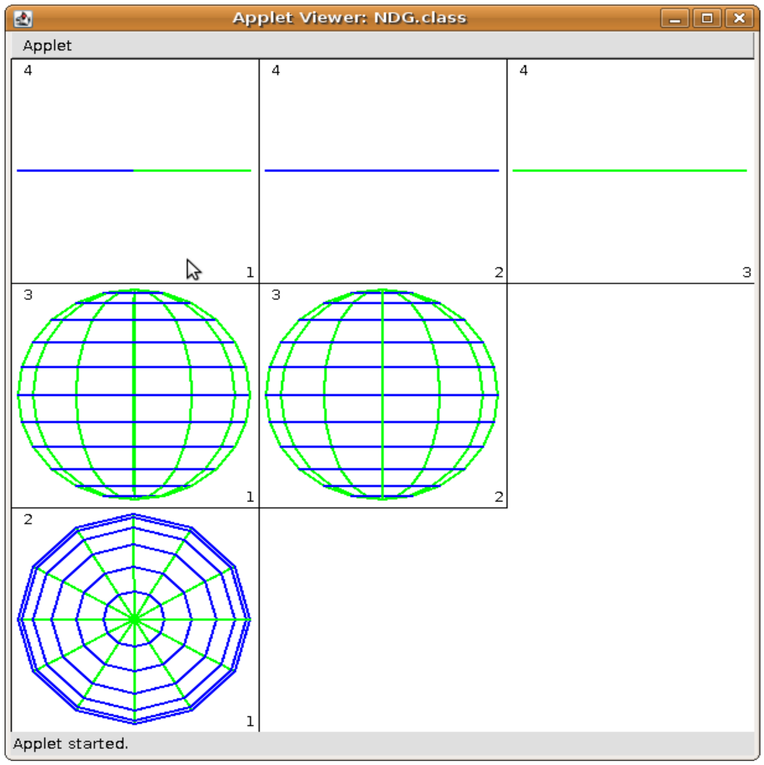

Figure 2: First equator of 4D Sphere. Red lines are absent, since $\alpha_0 = 0$, green lines correspond to the change of α_1, blue lines correspond to the change of α_2, The equator is a 3-D sphere in hyperplane (x_1, x_2, x_3). Coordinate $x_4 = 0$

hyperplane means that dimension of this plane is lower than dimension of all space under consideration. Point on 1-D line is 0-D hyperplane, with respect to that line and so on.

If $\alpha_1 = 0$ then

$$
\begin{aligned}
x_1 &= R \cos \alpha_0 \cos \alpha_2 \\
x_2 &= R \cos \alpha_0 \sin \alpha_2
\end{aligned}
$$

$$
\begin{aligned}
x_3 &= 0 \\
x_4 &= R\sin\alpha_0 \\
-\frac{\pi}{2} &< \alpha_0 < \frac{\pi}{2}, 0 \le \alpha_2 < 2\pi
\end{aligned}
$$

is another *equator* of 4-dimensional sphere in *hyperplane* (x_1, x_2, x_4). We can see this object og Figure 4.

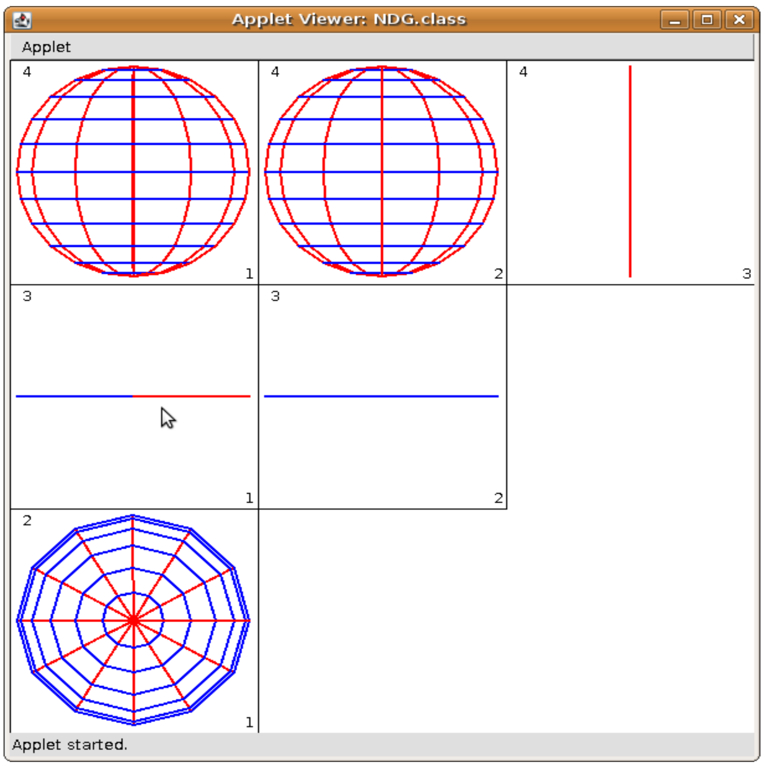

Figure 3: Second equator of 4D Sphere. Green lines are absent, since $\alpha_1 = 0$, red lines correspond to the change of α_0, blue lines correspond to the change of α_2, The equator is a 3-D sphere in hyperplane (x_1, x_2, x_4). Coordinate $x_3 = 0$

When $\alpha_0 = \pm\frac{\pi}{2}$ we have poles

$$
\begin{aligned}
x_1 &= 0 \\
x_2 &= 0
\end{aligned}
$$

$$x_3 = 0$$
$$x_4 = \pm R$$

Important to notice that in 3-D we have *axis of rotation* which comes through north and south poles, in 4-D we have *plane* (x_3, x_4), around which we are able to rotate the sphere. Really, looking at Eq.(9-13) you can see that change of longitude α_2 has no effect on (x_3, x_4). When longitude $\alpha_2 = 0$ we get *zero meridian*:

$$x_1 = R\cos\alpha_0\cos\alpha_1 \quad (15)$$
$$x_2 = 0 \quad (16)$$
$$x_3 = R\cos\alpha_0\sin\alpha_1 \quad (17)$$
$$x_4 = R\sin\alpha_0 \quad (18)$$
$$-\frac{\pi}{2} < \alpha_0, \alpha_1 < \frac{\pi}{2} \quad (19)$$

which is a 3-D half-sphere in subspace (x_1, x_3, x_4).

Intersection of the zero meridian with first equator ($\alpha_0 = \alpha_2 = 0$) is a circle

$$x_1 = R\cos\alpha_1, x_3 = R\sin\alpha_1, x_2 = x_4 = 0$$

in a plane (x_1, x_3). Intersection of the zero meridian with second equator ($\alpha_1 = \alpha_2 = 0$) is a circle

$$x_1 = R\cos\alpha_2, x_3 = R\sin\alpha_2, x_2 = x_4 = 0$$

in a plane (x_1, x_3).In 3-D space it was a point.

This 3-D meridian can be build with the receipt from previous section, as two subsequent rotations of one point, slightly modifying Eq. (7):

$$\begin{vmatrix} R\cos\alpha_0\cos\alpha_1 \\ 0 \\ R\cos\alpha_0\sin\alpha_1 \\ R\sin\alpha_0 \end{vmatrix} = \begin{vmatrix} \cos\alpha_1, & 0, & -\sin\alpha_1, & 0 \\ 0, & 1, & 0, & 0 \\ \sin\alpha_1, & 0, & \cos\alpha_1, & 0 \\ 0, & 0, & 0, & 1 \end{vmatrix} \begin{vmatrix} \cos\alpha_0, & 0, & 0, & -\sin\alpha_0 \\ 0, & 1, & 0, & 0 \\ 0, & 0, & 1, & 0 \\ \sin\alpha_0, & 0, & 0, & \cos\alpha_0 \end{vmatrix} \begin{vmatrix} R \\ 0 \\ 0 \\ 0 \end{vmatrix} \quad (20)$$

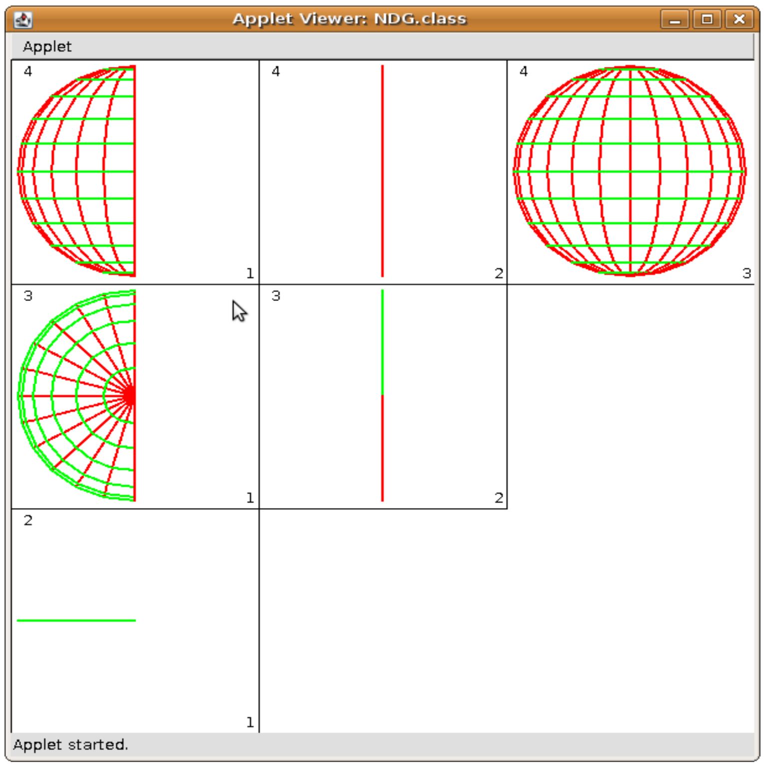

Figure 4: Zero meridian of 4D Sphere. Blue lines are absent, since $\alpha_2 = 0$, red lines correspond to the change of α_0, green lines correspond to the change of α_1, The meridian is a 3-D half-sphere in hyperplane (x_1, x_3, x_4). Coordinate $x_2 = 0$

First or rightmost rotation in Eq.8 is a rotation around the plane (x_2, x_3), since those coordinates are not affected, and the leftmost is a rotation around the plane (x_2, x_4). Similar to the 3-D case, if we *rotate this meridian around the plane* (x_3, x_4) we will get equation of 4-D sphere:

$$\begin{vmatrix} \cos\alpha_2, & -\sin\alpha_2, & 0, & 0 \\ \sin\alpha_2, & \cos\alpha_2, & 0, & 0 \\ 0, & 0, & 1, & 0 \\ 0, & 0, & 0, & 1 \end{vmatrix} \begin{vmatrix} R\cos\alpha_0\cos\alpha_1 \\ 0 \\ R\cos\alpha_0\sin\alpha_1 \\ R\sin\alpha_0 \end{vmatrix} = \begin{vmatrix} R\cos\alpha_0\cos\alpha_1\cos\alpha_2 \\ R\cos\alpha_0\cos\alpha_1\sin\alpha_2 \\ R\cos\alpha_0\sin\alpha_1 \\ R\sin\alpha_0 \end{vmatrix} \tag{21}$$

Result:to receive a 3-D surface of 4-D sphere we must take a single point and perform 3 rotations, around 3 planes. Let us ascend at one more space dimension.

3 5-dimensional sphere

Equation of the *body* of the sphere in 4-D Euclidean space (x_1, x_2, x_3, x_4) is:

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 \leq R^2 \tag{22}$$

It is a 5-D object. A *surface* of 5-D sphere

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 = R^2 \tag{23}$$

is a 4-dimensional, since x_1, x_2, x_3, x_4, x_5 are tied with equation (22), so only three of them are independent. It especially can be seen from parametric representation of sphere surface as

$$x_1 = R\cos\alpha_0\cos\alpha_1\cos\alpha_2\cos\alpha_3 \tag{24}$$
$$x_2 = R\cos\alpha_0\cos\alpha_1\cos\alpha_2\sin\alpha_3 \tag{25}$$
$$x_3 = R\cos\alpha_0\cos\alpha_1\sin\alpha_2 \tag{26}$$
$$x_4 = R\cos\alpha_0\sin\alpha_1 \tag{27}$$
$$x_5 = R\sin\alpha_0 \tag{28}$$
$$-\frac{\pi}{2} < \alpha_0, \alpha_1, \alpha_2 < \frac{\pi}{2}, 0 \leq \alpha_3 < 2\pi \tag{29}$$

where $\alpha_0, \alpha_1, \alpha_2$ are *three latitudes* and α_3 is *a longitude*.

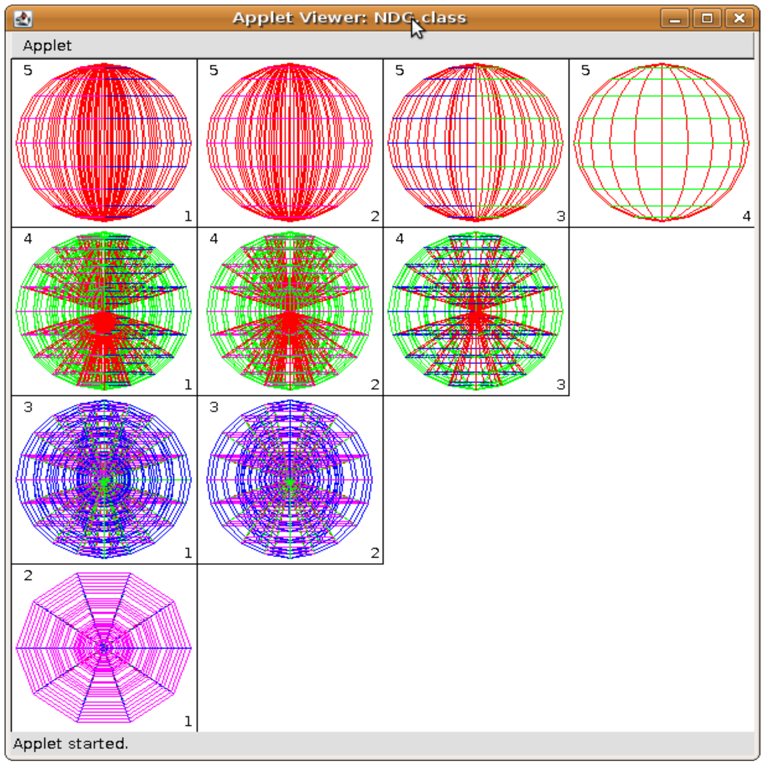

Figure 5: 5D Sphere. Red lines correspond to the change of α_0, green lines correspond to the change of α_1, blue lines correspond to the change of α_3, magenta lines correspond to the change of α_3

If $\alpha_0 = 0$ then

$$
\begin{aligned}
x_1 &= R\cos\alpha_1\cos\alpha_2\cos\alpha_3 \\
x_2 &= R\cos\alpha_1\cos\alpha_2\sin\alpha_3 \\
x_3 &= R\cos\alpha_1\sin\alpha_2 \\
x_4 &= R\sin\alpha_1 \\
x_5 &= 0 \\
-\frac{\pi}{2} < \alpha_1, \alpha_2 < \frac{\pi}{2}, 0 &\leq \alpha_3 < 2\pi
\end{aligned}
$$

is a 3-dimensional surface of 4-dimensional sphere, first *equator*, of 5-dimensional sphere in *hyperplane* (x_1, x_2, x_3, x_4).

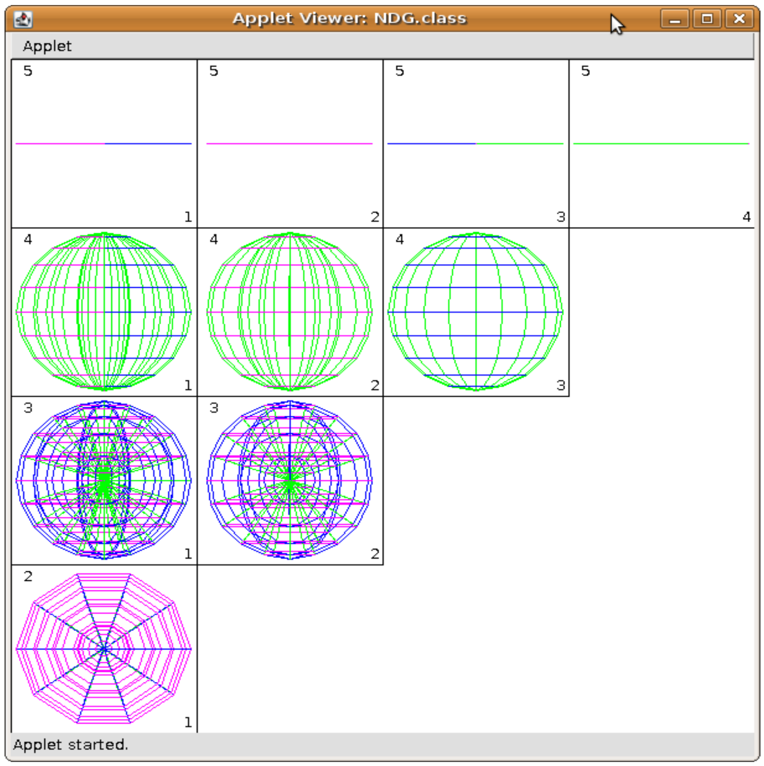

Figure 6: First equator of 5D Sphere. Red lines are absent, since $\alpha_0 = 0$, green lines correspond to the change of α_1, blue lines correspond to the change of α_3, magenta lines correspond to the change of α_3. Equator is a 4-D sphere in hyperplane (x_1, x_2, x_3, x_4)

If $\alpha_1 = 0$ then

$$\begin{aligned}
x_1 &= R\cos\alpha_0\cos\alpha_2\cos\alpha_3 \\
x_2 &= R\cos\alpha_0\cos\alpha_2\sin\alpha_3 \\
x_3 &= R\cos\alpha_0\sin\alpha_2 \\
x_4 &= 0 \\
x_5 &= R\sin\alpha_0
\end{aligned}$$

$$-\frac{\pi}{2} < \alpha_0, \alpha_2 < \frac{\pi}{2}, 0 \quad \leq \quad \alpha_3 < 2\pi$$

is another *equator* of 5-dimensional sphere in *hyperplane* (x_1, x_2, x_3, x_5).

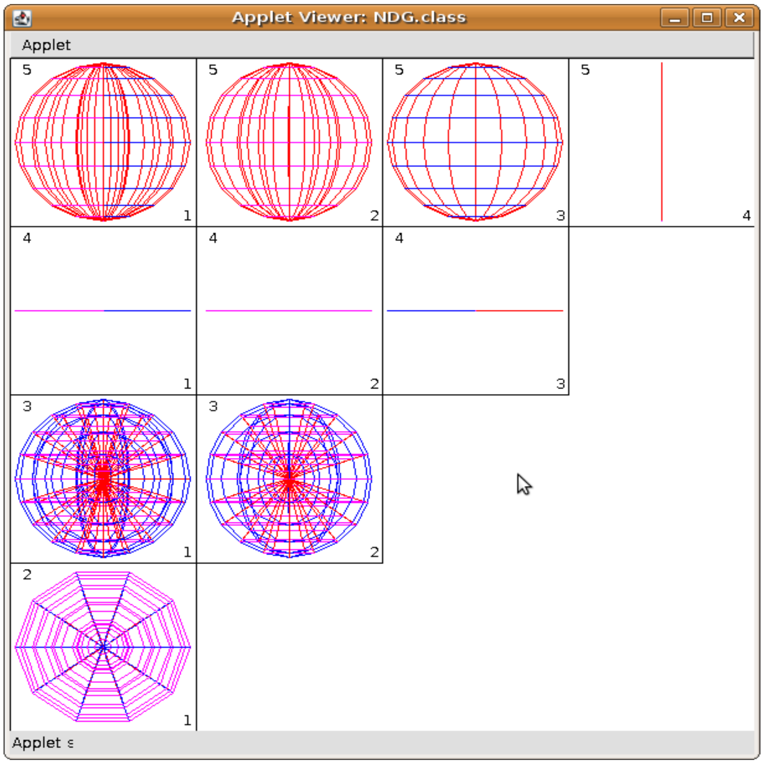

Figure 7: Second equator of 5D Sphere. Green lines are absent, since $\alpha_1 = 0$, red lines correspond to the change of α_0, blue lines correspond to the change of α_3, magenta lines correspond to the change of α_3. Equator is a 4-D sphere in hyperplane (x_1, x_2, x_3, x_5)

If $\alpha_2 = 0$ then

$$x_1 = R\cos\alpha_0\cos\alpha_1\cos\alpha_3 \quad (30)$$

$$x_2 = R\cos\alpha_0\cos\alpha_1\sin\alpha_3 \quad (31)$$

$$x_3 = 0 \quad (32)$$

$$x_4 = R\cos\alpha_0\sin\alpha_1 \quad (33)$$

$$
\begin{aligned}
x_5 &= R\sin\alpha_0 && (34)\\
-\frac{\pi}{2} < \alpha_0, \alpha_1, \alpha_2 < \frac{\pi}{2}, 0 &\leq \alpha_3 < 2\pi && (35)
\end{aligned}
$$

is another *equator* of 5-dimensional sphere in *hyperplane* (x_1, x_2, x_4, x_5).

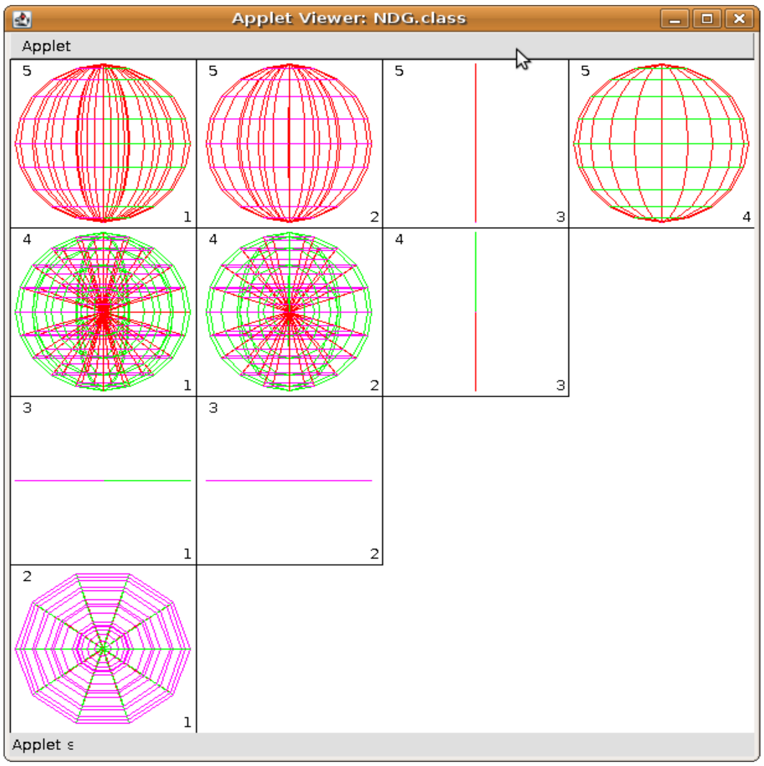

Figure 8: Third equator of 5D Sphere. Green lines are absent, since $\alpha_1 = 0$, red lines correspond to the change of α_0, blue lines correspond to the change of α_3, magenta lines correspond to the change of α_3. Equator is a 4-D sphere in hyperplane (x_1, x_2, x_3, x_5)

When $\alpha_0 = \pm\frac{\pi}{2}$ we have couple of poles

$$
\begin{aligned}
x_1 &= 0\\
x_2 &= 0\\
x_3 &= 0
\end{aligned}
$$

$$x_4 = 0$$
$$x_5 = \pm R$$

When longitude $\alpha_3 = 0$ we get *zero meridian*

$$x_1 = R\cos\alpha_0\cos\alpha_1\cos\alpha_2 \quad (36)$$
$$x_2 = 0 \quad (37)$$
$$x_3 = R\cos\alpha_0\cos\alpha_1\sin\alpha_2 \quad (38)$$
$$x_4 = R\cos\alpha_0\sin\alpha_1 \quad (39)$$
$$x_5 = R\sin\alpha_0 \quad (40)$$
$$-\frac{\pi}{2} < \alpha_0, \alpha_1, \alpha_2 < \frac{\pi}{2} \quad (41)$$

which is a 4-D half sphere in subspace (x_1, x_3, x_4, x_5).

Intersections of this meridian with equators are 3-D spheres, fact can be easily checked by the reader. Similar to the 4-D case, if we *rotate this meridian around the plane* (x_3, x_4, x_5) we will get equation of 5-D sphere:

$$\begin{vmatrix} \cos\alpha_3, & -\sin\alpha_3, & 0, & 0, & 0 \\ \sin\alpha_3, & \cos\alpha_3, & 0, & 0, & 0 \\ 0, & 0, & 1, & 0, & 0 \\ 0, & 0, & 0, & 1, & 0 \\ 0. & 0, & 0, & 0, & 1 \end{vmatrix} \begin{vmatrix} R\cos\alpha_0\cos\alpha_1\cos\alpha_2 \\ 0 \\ R\cos\alpha_0\cos\alpha_1\sin\alpha_2 \\ R\cos\alpha_0\sin\alpha_1 \\ R\sin\alpha_0 \end{vmatrix} = \begin{vmatrix} R\cos\alpha_0\cos\alpha_1\cos\alpha_2\cos\alpha_3 \\ R\cos\alpha_0\cos\alpha_1\cos\alpha_2\sin\alpha_3 \\ R\cos\alpha_0\cos\alpha_1\sin\alpha_2 \\ R\sin\alpha_0\sin\alpha_1 \\ R\sin\alpha_0 \end{vmatrix} \quad (42)$$

And meridian itself can be provided by rotations of single point, just as it explained in 4-D sphere section.
Finally, answering the question in the article's title, we provide a table of some properties of the spheres in different space dimensions:

4 Conclusion

In this article we tried to demonstrate how analysis of elementary formulas unveils the richness of properties of a simple object like sphere, and how

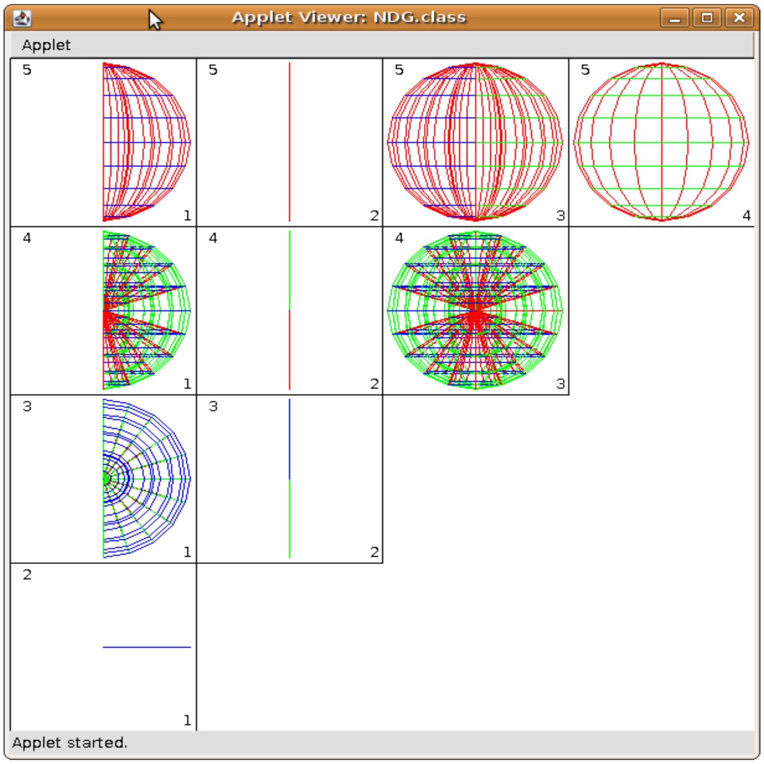

Figure 9: Zero meridian of 5D Sphere. Magenta lines are absent, since $\alpha_3 = 0$, Red lines correspond to the change of α_0, green lines correspond to the change of α_1, blue lines correspond to the change of α_2. Meridian is a 4-D half sphere in hyperplane (x_1, x_3, x_4, x_5)

	Equator	Meridian	Pole
3-D	1	1	2
4-D	2	1	2
5-D	3	1	2

Table 1: Elements of the sphere

those properties evolve with the raise of dimensions of space. We also produced a method of displaying of multidimensional object, which expands standard technical drafting. More examples of this approach reader can find at www.asymptotus.com.

References

[1] G.A,Korn,T.M.Korn *Mathematical handbook for scientists and engineers*,Dover, NY, 2000